

Complete mappings and Carlitz rank

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Abstract

The well-known Chowla and Zassenhaus conjecture, proven by Cohen in 1990, states that for any $d \geq 2$ and any prime $p > (d^2 - 3d + 4)^2$ there is no complete mapping polynomial in $\mathbb{F}_p[x]$ of degree d .

For arbitrary finite fields \mathbb{F}_q , we give a similar result in terms of the Carlitz rank of a permutation polynomial rather than its degree. We prove that if $n < \lfloor q/2 \rfloor$, then there is no complete mapping in $\mathbb{F}_q[x]$ of Carlitz rank n of small linearity. We also determine how far permutation polynomials f of Carlitz rank $n < \lfloor q/2 \rfloor$ are from being complete, by studying value sets of $f + x$. We provide examples of complete mappings if $n = \lfloor q/2 \rfloor$, which shows that the above bound cannot be improved in general.

Keywords: Permutation polynomials, complete mappings, Carlitz rank, value sets of polynomials

Mathematical Subject Classification: 11T06

1 Introduction

For any prime power q let \mathbb{F}_q be the finite field of q elements. A polynomial $f(x) \in \mathbb{F}_q[x]$ is called a *permutation polynomial* if it induces a bijection from \mathbb{F}_q to \mathbb{F}_q .

A polynomial $f(x) \in \mathbb{F}_q[x]$ is a *complete mapping polynomial* (or a complete mapping) if both $f(x)$ and $f(x) + x$ are permutation polynomials of \mathbb{F}_q . These polynomials were introduced by Mann in 1942, [12]. A detailed study of complete mapping polynomials over finite fields was carried out by Niederreiter and Robinson (1982, [14]). Complete mappings are pertinent to the construction of mutually orthogonal Latin squares, which can be used for the design of agricultural experiments, see for example [10]. Also due to other recently emerged applications such as check-digit systems [17, 18] and the construction of cryptographic functions [13, 19], complete mappings have attracted considerable attention, see also [8, 9, 15, 21, 22, 23, 24].

By a well-known result of Carlitz (1953), all permutation polynomials over \mathbb{F}_q with $q \geq 3$ can be generated by linear polynomials $ax + b$, $a, b \in \mathbb{F}_q$, $a \neq 0$, and *inversions* $x^{q-2} = \begin{cases} 0, & x = 0, \\ x^{-1}, & x \neq 0, \end{cases}$ see [2] or [11, Theorem 7.18]. Consequently, as pointed out in [4], any permutation f of \mathbb{F}_q can be represented by a polynomial of the form

$$P_n(a_0, a_1, \dots, a_{n+1}; x) = (\dots((a_0x + a_1)^{q-2} + a_2)^{q-2} \dots + a_n)^{q-2} + a_{n+1}, \quad (1)$$

where $a_i \neq 0$, for $i = 0, 2, \dots, n$. Note that this representation is not unique, and n is not necessarily minimal. Accordingly the authors of [1] define the *Carlitz rank* of a permutation polynomial f over \mathbb{F}_q to be the smallest integer $n \geq 0$ satisfying $f = P_n$ for a permutation P_n of the form (1), and denote it by $Crk(f)$. In other words, for $q \geq 4$, $Crk(f) = n$ if f is a composition of at least n inversions x^{q-2} and n or $n + 1$ linear polynomials (depending on a_{n+1} being zero or not). This concept, introduced in the last decade, has already found interesting applications in diverse areas, see [5, 7, 16].

The following theorem states the well-known conjecture of Chowla and Zassenhaus (1968) [3], which was proven by Cohen [6] in 1990.

Theorem A. If $d \geq 2$ and $p > (d^2 - 3d + 4)^2$, then there is no complete mapping polynomial of degree d over \mathbb{F}_p .

Note that Cohen's theorem is not true for arbitrary finite fields without further restrictions. For example, for any $0 \neq a \in \mathbb{F}_{p^r}$ with $a^{(p^r-1)/(p-1)} \neq (-1)^r$ it is easy to see that ax^p is a complete mapping.

Since the Carlitz rank of a permutation polynomial f over \mathbb{F}_q is an invariant of f , a natural question to ask is whether a non-existence result, similar to that stated in Theorem A, can be obtained in terms of the Carlitz rank.

We define the *linearity* $\mathcal{L}(f)$ of a polynomial f over \mathbb{F}_q by

$$\mathcal{L}(f) = \max_{a, b \in \mathbb{F}_q} |\{c \in \mathbb{F}_q : f(c) = ac + b\}|.$$

Note that polynomials of large linearity are highly predictable and thus unsuitable in cryptography.

In this paper we show, see Theorem 1 below, that for any $n < \lfloor q/2 \rfloor$, there is no complete mapping polynomial of Carlitz rank n and linearity $\mathcal{L}(f) < \lfloor (q+5)/2 \rfloor$.

We also answer the following two questions that immediately arise. Firstly one wonders how far the non-complete mapping f in the above setting is from being complete. This question can be quantified by considering the number $|V_{f+x}|$ of elements in the image of the polynomial $f+x$. Theorem 3 presents bounds for $|V_{f+x}|$. Secondly one would ask if the bound $q > 2n+1$ can be improved. This is not possible in general, see Example 2 below.

2 Preliminaries

Let $f(x)$ be a permutation polynomial over \mathbb{F}_q . Suppose that f has a representation P_n as in (1) for $n \geq 1$. We follow the notation of [20] and put

$$f(x) = P_n(a_0, a_1, \dots, a_{n+1}; x).$$

Since we are interested in complete mapping polynomials, the value of a_{n+1} is irrelevant. Also, by using the substitution $x \mapsto x - a_0^{-1}a_1$, we see that the size of the value set of $f(x) + x$ does not depend on a_1 . Therefore we may restrict ourselves to the case $a_1 = a_{n+1} = 0$. We relabel the coefficients accordingly, as $c_0 = a_0$, $c_i = a_{i+1}$ for $i = 1, \dots, n-1$, and use the notation

$$f(x) = P_n(c_0, \dots, c_{n-1}; x) =: P_n(x). \quad (2)$$

The representation of a permutation f as in (1) (or in (2)) enables approximation of f by a fractional linear transformation R_n as described below.

Following the terminology of [1], the *n*th convergent $R_n(x)$ can be associated to f , which is defined as

$$R_n(x) = \frac{\alpha_{n-1}x + \beta_{n-1}}{\alpha_n x + \beta_n}, \quad (3)$$

where

$$\alpha_k = c_{k-1}\alpha_{k-1} + \alpha_{k-2} \quad \text{and} \quad \beta_k = c_{k-1}\beta_{k-1} + \beta_{k-2},$$

for $k \geq 2$ and $\alpha_0 = 0$, $\alpha_1 = c_0$, $\beta_0 = 1$, $\beta_1 = 0$.

The set of *poles* \mathbf{O}_n is defined as

$$\mathbf{O}_n = \{x_i : x_i = \frac{-\beta_i}{\alpha_i}, i = 1, \dots, n\} \subset \mathbb{F}_q \cup \{\infty\},$$

where the elements of \mathbf{O}_n are not necessarily distinct. We note that

$$f(c) = P_n(c) = R_n(c) \quad \text{for } c \in \mathbb{F}_q \setminus \mathbf{O}_n. \quad (4)$$

3 A non-existence result

In this section we show that any complete mapping must have either high Carlitz rank or high linearity.

Theorem 1. *If $f(x)$ is a complete mapping of \mathbb{F}_q , then we have either*

$$\mathcal{L}(f) \geq \left\lfloor \frac{q+5}{2} \right\rfloor$$

or

$$\text{Crk}(f) \geq \left\lfloor \frac{q}{2} \right\rfloor.$$

Proof. Let $f(x)$ be of the form (2) with $n = \text{Crk}(f)$ and put $F(x) = f(x) + x$. For $n = 0$ we have $\mathcal{L}(f) = q$. Hence, we may assume $n \geq 1$.

If $\alpha_n = 0$, then $R_n(x)$ defined by (3) is a polynomial of degree 1 with $R_n(c) = f(c)$ for all $c \in \mathbb{F}_q \setminus \mathbf{O}_n$ by (4) and thus $\mathcal{L}(f) \geq q - n + 1$. Since otherwise the result is trivial, we may assume $n \leq \lfloor q/2 \rfloor - 1$ and thus $\mathcal{L}(f) \geq q + 2 - \lfloor q/2 \rfloor = \lfloor (q+5)/2 \rfloor$.

Now we assume $\alpha_n \neq 0$.

We note that the first pole x_1 is 0, since $\beta_1 = 0$. Observe that

$$F(c) = R_n(c) + c = \frac{\alpha_n c^2 + (\alpha_{n-1} + \beta_n)c + \beta_{n-1}}{\alpha_n c + \beta_n} \quad (5)$$

for any $c \in \mathbb{F}_q \setminus \mathbf{O}_n$. It is also easy to show that

$$\alpha_n \beta_{n-1} - \alpha_{n-1} \beta_n = (-1)^{n-1} c_0, \quad n \geq 1. \quad (6)$$

First we assume that q is odd.

For any $u \in \mathbb{F}_q$ we study the quadratic equation

$$R_n(x) + x = u + (\alpha_{n-1} - \beta_n) \alpha_n^{-1}, \quad (7)$$

that is,

$$x^2 + (2\alpha_n^{-1} \beta_n - u)x + ((-1)^{n-1} c_0 + \beta_n^2 - u \alpha_n \beta_n) \alpha_n^{-2} = 0 \quad (8)$$

by (5) and (6). This equation has at most two different solutions $c \in \mathbb{F}_q \setminus \{x_n\}$ and we have exactly two solutions if its discriminant

$$D_u = u^2 + 4(-1)^n c_0 \alpha_n^{-2} \quad (9)$$

is a square in \mathbb{F}_q^* . Note that

$$\frac{1 + \eta(D_u)}{2} = \begin{cases} 1, & D_u \text{ is a square in } \mathbb{F}_q^*, \\ 0, & D_u \text{ is a nonsquare in } \mathbb{F}_q^*, \\ 1/2, & D_u = 0, \end{cases}$$

where η is the quadratic character of \mathbb{F}_q . Moreover, either $D_u = 0$ for two values of u , that is, $(-1)^{n-1} c_0$ is a square, or there is no value u with $D_u = 0$. Hence, the number N of the elements $u \in \mathbb{F}_q$ for which D_u is a square in \mathbb{F}_q^* can be expressed as

$$\begin{aligned} N &= \frac{1}{2} \sum_{u \in \mathbb{F}_q, D_u \neq 0} (1 + \eta(D_u)) = -\frac{1 + \eta((-1)^{n-1} c_0)}{2} + \frac{1}{2} \sum_{u \in \mathbb{F}_q} (1 + \eta(D_u)) \\ &= \frac{q - 1 - \eta((-1)^{n-1} c_0)}{2} + \frac{1}{2} \sum_{u \in \mathbb{F}_q} \eta(D_u) = \frac{q - 2 - \eta((-1)^{n-1} c_0)}{2}, \end{aligned}$$

by [11, Theorem 5.48].

Now assume that F is a permutation. Then at least one of these two solutions must be a pole $c \in \mathbf{O}_n \setminus \{x_n\}$. Hence,

$$n \geq \frac{q - \eta((-1)^{n-1} c_0)}{2} \geq \frac{q - 1}{2}.$$

For even q we can argue similarly. Note that a quadratic equation $x^2 + ax + b$ has exactly two solutions whenever $a \neq 0$ and $\text{Tr}(a^{-2}b) = 0$, where Tr denotes the absolute trace of \mathbb{F}_q , see [11, Theorem 2.25]. We have to determine the number N of u such that (8) has two solutions in \mathbb{F}_q , that is, the number of $u \neq 0$ with

$$0 = \text{Tr} \left(\frac{\alpha_n \beta_n u + \beta_n^2 + c_0}{\alpha_n^2 u^2} \right) = \text{Tr} \left(\frac{\beta_n}{\alpha_n u} + \frac{\beta_n + c_0^{q/2}}{\alpha_n u} \right) = \text{Tr} \left(\frac{c_0^{q/2}}{\alpha_n u} \right). \quad (10)$$

Since $u \mapsto u^{-1}$ is a bijection of \mathbb{F}_q^* and Tr is 2-to-1 on \mathbb{F}_q , we get $N = q/2 - 1$. Hence, if F is a permutation, then \mathbf{O}_n contains at least $n \geq N + 1 = \frac{q}{2}$ different poles and the result follows. \square

Remark. Note that complete mappings of high linearity, that is, polynomials $f(x)$ with n th convergent $R_n(x)$ and $\alpha_n = 0$ (or $x_n = \infty$) are not suitable for cryptographic applications. Hence, in the following we focus on the case $\alpha_n \neq 0$ (or $x_n \neq \infty$). Note that $\alpha_1\alpha_2 \neq 0$ and thus ∞ is not a pole if $n = 1$ or $n = 2$.

Now we provide examples of complete mappings of Carlitz rank $n = \lfloor q/2 \rfloor$ with $\mathcal{L}(f) < \lfloor (q+5)/2 \rfloor$.

Example 2. *It is easy to check that $f(x) = \gamma(x^4+1) + \gamma^{-1}(x^2+x) \in \mathbb{F}_8[x]$ is a complete mapping of $\mathbb{F}_8 = \mathbb{F}_2(\gamma)$, where γ is a root of the polynomial x^3+x+1 which is irreducible over \mathbb{F}_2 . As a polynomial of degree 4 its linearity is at most 4 and by Theorem 1 its Carlitz rank is at least 4. Verifying*

$$f(c) = (((\gamma c)^6 + 1)^6 + \gamma^{-3})^6 + 1)^6, \quad c \in \mathbb{F}_8,$$

we see that $\text{Crk}(f) = 4$ and Theorem 1 is in general tight in the case of even q .

Analogously, $f(x) = x^4 - x^3 + 3x^2 - x + 1 \in \mathbb{F}_7[x]$ satisfies

$$f(c) = (((c^5 + 3)^5 + 3)^5, \quad c \in \mathbb{F}_7,$$

and has Carlitz rank 3. Hence, the bound of Theorem 1 is attained for odd q , as well.

Many similar examples lead the authors to believe that there is a complete mapping of \mathbb{F}_q of Carlitz rank $n = \lfloor q/2 \rfloor$ and small linearity for infinitely many prime powers $q \geq 7$. This can be checked for $7 \leq q \leq 25$.

4 The size of V_{f+x}

In this section we study the set $V_{f+x} = \{f(\delta) + \delta : \delta \in \mathbb{F}_q\}$ for any f satisfying (4) with $\alpha_n \neq 0$. Theorem 1 implies that if $n < \lfloor q/2 \rfloor$, we have $|V_{f+x}| < q$. Here we aim to determine how large the gap between q and $|V_{f+x}|$ is. Theorem 3 below shows that $q - |V_{f+x}| \geq (q - 2 \text{Crk}(f) - 1)/2$, that is, it is large if the Carlitz rank of f is small, as one would expect. We present the result in a slightly more general form.

Theorem 3. *For $\alpha_{n-1}, \beta_{n-1}, \alpha_n, \beta_n \in \mathbb{F}_q$ with $\alpha_n \neq 0$ and $\alpha_{n-1}\beta_n - \alpha_n\beta_{n-1} \neq 0$, let F be any self-mapping of \mathbb{F}_q satisfying*

$$F(c) = \frac{\alpha_{n-1}c + \beta_{n-1}}{\alpha_nc + \beta_n} + c \tag{11}$$

for at least $q - n$ different $c \in \mathbb{F}_q$. Then we have

$$\left\lceil \frac{q - n}{2} \right\rceil \leq |V_F| \leq \min \left\{ n + \left\lfloor \frac{q + 1}{2} \right\rfloor, q \right\}.$$

Proof. Consider the set S of elements $c \in \mathbb{F}_q$ satisfying (11), which has cardinality $|S| \geq q - n$. At most two different elements of S can have the same value u since $F(c) = u$ is a quadratic equation in c because of the conditions on $\alpha_{n-1}, \beta_{n-1}, \alpha_n, \beta_n$. Therefore, $|V_F| \geq (q - n)/2$. Now the elements of $\mathbb{F}_q \setminus S$ can attain at most n different values of F . If q is odd, the discriminant D_u of $F(c) = u$ is a quadratic polynomial in u and is 0 for at most two different values $u \in V_F$. For these two possible u we have exactly one solution c of $F(c) = u$. For all other u we have either two or no solutions. Hence, the value set of $\frac{\alpha_{n-1}x + \beta_{n-1}}{\alpha_n x + \beta_n} + x$ contains at most $(q + 1)/2$ elements and we get $|V_F| \leq n + (q + 1)/2$. If q is even, the quadratic equation $F(c) = u$ has a unique solution for exactly one u and two or no solutions otherwise. Hence, we get similarly $|V_F| \leq n + q/2$. \square

For the special cases $n = 1$ and $n = 2$ one can provide exact formulas for $|V_{f+x}|$.

Proposition 4. *The size of the value set V_F of the polynomial*

$$F(x) = (c_0 x)^{q-2} + x \in \mathbb{F}_q[x],$$

$q > 2$, with $c_0 \neq 0$ is

$$|V_F| = \begin{cases} (q + 1 + \eta(c_0) - \eta(-c_0))/2, & q \text{ odd}, \\ q/2, & q \text{ even}, \end{cases}$$

where η denotes the quadratic character of \mathbb{F}_q .

Proof. We start with odd q . We have $F(0) = 0 = F(\pm(-c_0)^{-1/2})$ and thus $F(c) = 0$ is attained for $2 + \eta(-c_0)$ different $c \in \mathbb{F}_q$. The discriminant

$$D_u = u^2 - 4c_0^{-1}$$

of $x^2 - ux + c_0^{-1}$ has no zeros if c_0 is a non-square. If c_0 is a square, for the two zeros of D_u there is a unique solution $c = u/2$ of $F(c) = u$. For the remaining u there are two or no solutions of $F(c) = u$. Collecting everything we get the result.

For even q we have $F(0) = F(c_0^{-q/2}) = 0$ and no further zeros of F . For all $u \neq 0$ there are either two or no solutions of $F(c) = u$ and we get the result. \square

Proposition 5. *The size of the value set of $F(x) = ((c_0x)^{q-2} + c_1)^{q-2} + x$, $q > 2$, with $c_0, c_1, 4c_0 + 1, c_0 + 4 \neq 0$ is*

$$|V_F| = \begin{cases} \frac{q+2-\eta(4c_0+1)-\eta(c_0^2+4c_0)+\eta(-c_0)}{2}, & c_0 \neq -1, \\ \frac{q-\eta(-3)}{2}, & c_0 = -1, \end{cases}$$

if q is odd. For even q and $c_0, c_1 \neq 0$, we get

$$|V_F| = \frac{q}{2} + \begin{cases} \text{Tr}(c_0) + \text{Tr}(c_0^{-1}), & c_0 \neq 1, \\ \text{Tr}(1) - 1, & c_0 = 1, \end{cases}$$

where Tr is the absolute trace of \mathbb{F}_q and we identify \mathbb{F}_2 with the integers $\{0, 1\}$.

Proof. Note that $\mathbf{O}_2 = \{0, -(c_0c_1)^{-1}\}$. We have $F(0) = c_1^{-1}$ and

$$F(-(c_0c_1)^{-1}) = -(c_0c_1)^{-1}.$$

Note that both values coincide if $c_0 = -1$. (7) simplifies to $R_2(x) + x = u + c_1^{-1} - (c_0c_1)^{-1}$. Hence, we get $R_2(c) + c = F(0)$ if $u = (c_0c_1)^{-1} =: u_1$ and $R_2(c) + c = F(-(c_0c_1)^{-1})$ if $u = -c_1^{-1} =: u_2$.

Again we deal with odd q first.

By (9) we get the discriminants

$$D_{u_1} = (4c_0 + 1)(c_0c_1)^{-2} \quad \text{and} \quad D_{u_2} = (c_0 + 4)c_0(c_0c_1)^{-2}.$$

Hence there are $1 + \eta(4c_0 + 1)$ additional c with $R_2(c) + c = F(0)$ and $1 + \eta((c_0 + 4)c_0)$ additional c with $R_2(c) + c = F(-(c_0c_1)^{-1})$. Now verify that there is a u , namely $u = (1 - c_0)(c_0c_1)^{-1}$, such that $x = 0$ is a solution of (8). If $c_0 = -1$, $x = 0$ is the unique solution for this u . However, for $x = -(c_0c_1)^{-1}$ there is no such u . Finally, there are $1 + \eta(-c_0)$ values u with $D_u = 0$ such that (8) has a unique solution. Altogether we have

$$4 + \eta(-c_0) + \frac{q - 6 - \eta(4c_0 + 1) - \eta((c_0 + 4)c_0) - \eta(-c_0)}{2}$$

values in V_F if $c_0 \neq -1$ and the first result follows. For $c_0 = -1$ we get $|V_F| = 2 + \frac{q-4-\eta(-3)}{2}$.

Now we consider even q . By (10) and

$$\text{Tr}\left(\frac{c_0^{q/2}}{\alpha_2 u_1}\right) = \text{Tr}(c_0) \quad \text{and} \quad \text{Tr}\left(\frac{c_0^{q/2}}{\alpha_2 u_2}\right) = \text{Tr}(c_0^{-1})$$

the number of c with $F(c) = F(0)$ (including $c = 0$) is $3 - 2Tr(c_0)$ and the number of c with $F(c) = F((c_0c_1)^{-1})$ is $3 - 2Tr(c_0^{-1})$. For $u = 0$ there is a unique solution $x \neq 0$ of (8) if $c_0 \neq 1$. Moreover, $x = 0$ is a solution of (8) for one u which has already been counted above. Hence, we get

$$|V_F| = 4 + \frac{q - 8 + 2Tr(c_0) + 2Tr(c_0^{-1})}{2}$$

if $c_0 \neq 1$ and the result follows.

If $c_0 = 1$ we have $F(0) = F((c_0c_1)^{-1}) = c_1^{-1}$ and c_1^{-1} is attained $4 - 2Tr(c_0)$ times. Moreover, the u with unique solution (8) corresponds to the solution $x = 0$. Hence we get

$$|V_F| = 1 + \frac{q - 4 + 2Tr(1)}{2}$$

and the result follows. □

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